Division of Strength of Materials and Structures

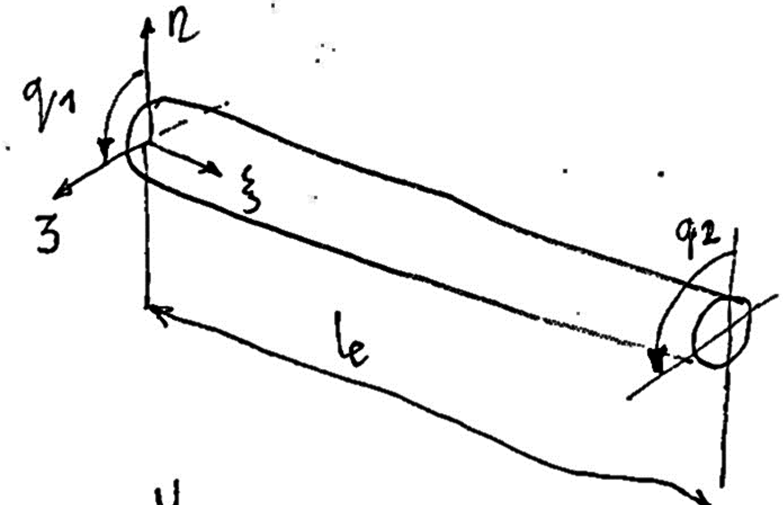
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

Lecture 8B. Torsion bar finite element

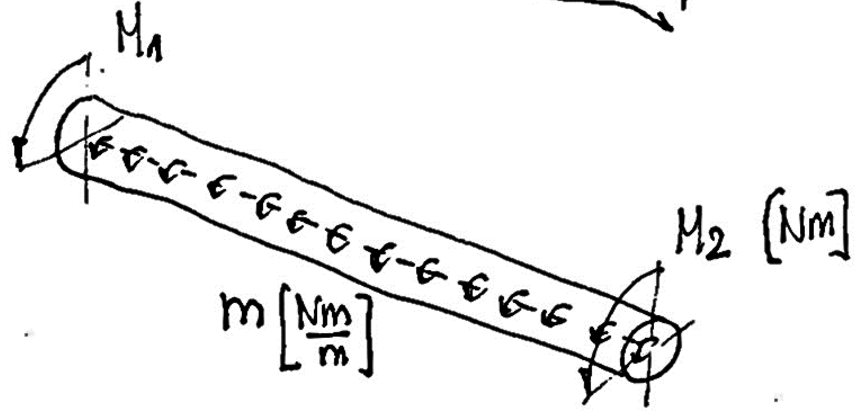
04.2025

Finite element of a rod loaded with a torsional moment



q_1, q_2 - rotations (twist angles)

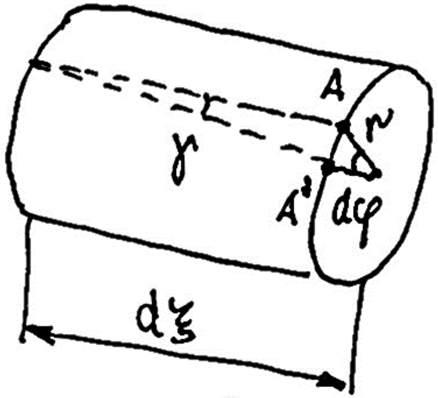
Load:



M- torque

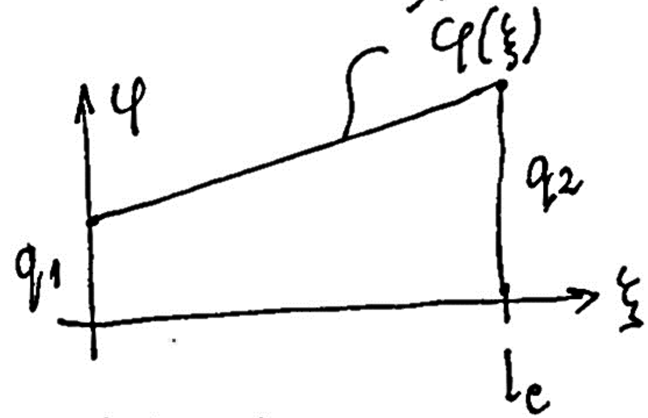
Torque per unit length

Torsion bar finite element - deformations



$$AA' \cong \gamma \cdot d\xi = r\,d\varphi$$

$$\gamma = r\,\frac{d\varphi}{d\xi}$$



Shape functions:
$$\boxed{\begin{aligned} N_1(\xi) &= 1 - \frac{\xi}{l_e} \\ N_2(\xi) &= \frac{\xi}{l_e} \end{aligned}}$$

$$\varphi(\xi) = N_1(\xi)\cdot q_1 + N_2(\xi)\cdot q_2 = \underset{1\times 2}{\lfloor N \rfloor}\cdot \underset{2\times 1}{\{q\}_e}$$

$$\frac{d\varphi}{d\xi} = \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \underset{2\times 1}{\{q\}_e} = \left\lfloor -\frac{1}{l_e}, \frac{1}{l_e} \right\rfloor \cdot \underset{2\times 1}{\{q\}_e}$$

Torsion bar finite element – elastic strain energy

$$U_e = \frac{1}{2}\int_{\Omega_e} \tau \cdot \gamma \, d\Omega_e = \frac{1}{2}\int_{\Omega_e} G \cdot \gamma^2 \, d\Omega_e = \frac{1}{2}\int_{\Omega_e} G\, r^2 \left(\frac{d\varphi}{d\xi}\right)^2 d\Omega_e =$$

Hook’s law:

$$\tau = G \cdot \gamma = \frac{E}{2(1+\nu)} \cdot \gamma$$

$$= \frac{1}{2}\int_0^{l_e} G \, \frac{d\varphi}{d\xi} \cdot \frac{d\varphi}{d\xi} \cdot \underbrace{\boxed{\int_A r^2 \, dA}}_{\text{polar moment of inertia } J_s} d\xi =$$

$$J_s = \frac{\pi d^4}{32}$$ For a circular cross section

Torsion bar finite element – stiffness matrix

$$= \frac{1}{2}\int_0^{l_e} GJ_s \cdot \frac{d\varphi}{d\xi} \cdot \frac{d\varphi}{d\xi}\, d\xi =$$

$$\underset{1\times 2}{\lfloor q \rfloor_e} \cdot \begin{Bmatrix} \frac{dN_1}{d\xi} \\ \frac{dN_2}{d\xi} \end{Bmatrix} \qquad \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \underset{2\times 1}{\{q\}_e}$$

$$= \frac{1}{2} \lfloor q \rfloor_e \cdot [k]_e \cdot \{q\}_e \quad ;$$

where:

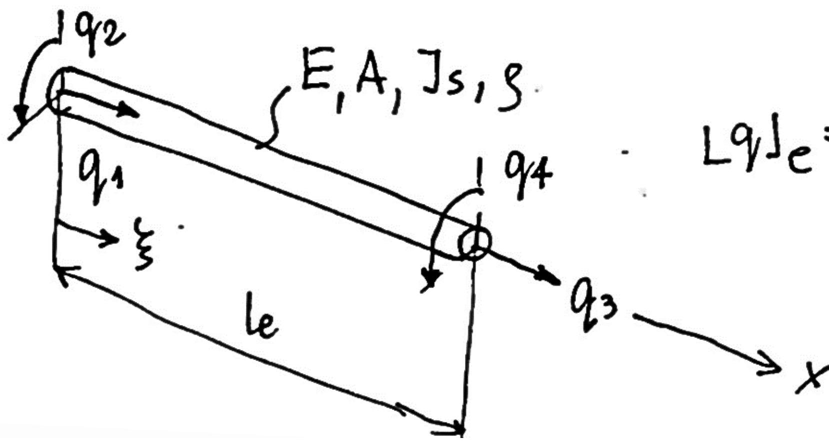
$$\underset{2\times 2}{[k]_e} = GJ_s \begin{bmatrix} \int_0^{l_e} \frac{dN_1}{d\xi}\frac{dN_1}{d\xi} d\xi & \int_0^{l_e} \frac{dN_1}{d\xi}\frac{dN_2}{d\xi} d\xi \\ \int_0^{l_e} \frac{dN_2}{d\xi}\frac{dN_1}{d\xi} d\xi & \int_0^{l_e} \frac{dN_2}{d\xi}\frac{dN_2}{d\xi} d\xi \end{bmatrix} = \boxed{\frac{GJ_s}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

Torsion bar finite element – potential energy of the load

$$W_e = \underset{1\times 2}{\lfloor F \rfloor_e} \cdot \{q\}_e + M_1 \cdot q_1 + M_2 \cdot q_2$$

$$\underset{1\times 2}{\lfloor F \rfloor_e} = \left\lfloor \int_0^{l_e} m(\xi) \cdot N_1(\xi)\, d\xi , \int_0^{l_e} m(\xi) \cdot N_2(\xi)\, d\xi \right\rfloor$$

Element loaded with axial force and torsional moment



$$\lfloor q \rfloor_e = \lfloor q_1, q_2, q_3, q_4 \rfloor$$

Element loaded with axial force:

$$\underset{2\times 2}{[k]_A} = \frac{EA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\lfloor q \rfloor_A = \lfloor q_1, q_3 \rfloor \quad \text{(translational D.O.F)}$$

$$\underset{4\times 4}{[k]_A^*} = \frac{EA}{l_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Element loaded with a torsional moment:

$$\underset{2\times 2}{[k]_T} = \frac{GJ_s}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\lfloor q \rfloor_T = \lfloor q_2, q_4 \rfloor \quad \text{(rotational D.O.F)}$$

$$\underset{4\times 4}{[k]_T^*} = \frac{GJ_s}{l_e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

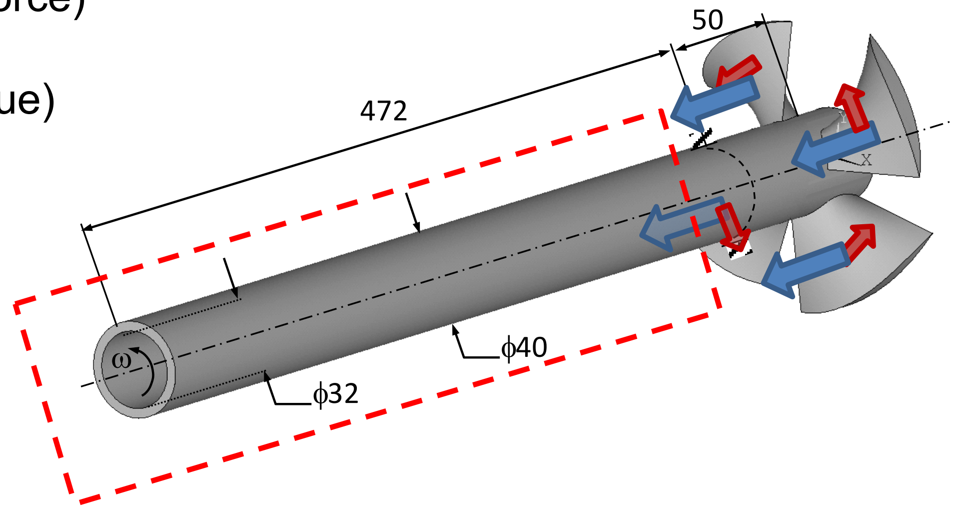
Element loaded with axial force and torsional moment – stiffness matrix

$$\underset{4\times4}{[k]_e} = \underset{4\times4}{[k]^*_A} + \underset{4\times4}{[k]^*_T} = \frac{1}{l_e}\begin{bmatrix} EA & 0 & -EA & 0 \\ 0 & GJ_s & 0 & -GJ_s \\ -EA & 0 & EA & 0 \\ 0 & -GJ_s & 0 & GJ_s \end{bmatrix}$$

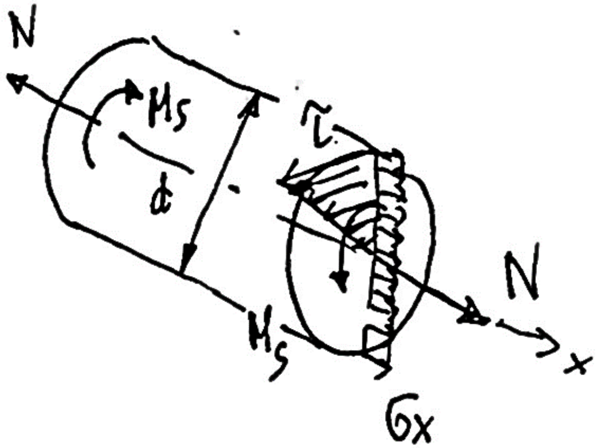
Example: An element loaded with an axial force and a torsional moment used in the FEM model of a ship's propeller shaft.

lift force (axial force)

drag force (torque)



Stress state in the FEM model of the ship's propulsion shaft



$$\sigma_x = E \cdot \varepsilon_x = E \cdot \frac{du}{d\xi} =$$

$$= E \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \begin{Bmatrix} q_1 \\ q_3 \end{Bmatrix}_A =$$

$$= E \, \frac{q_3 - q_1}{l_e}$$

$$\frac{dN_1}{d\xi} = -\frac{1}{l_e} \quad , \quad \frac{dN_2}{d\xi} = \frac{1}{l_e}$$

$$\tau(r) = G \cdot \gamma(r) = G \cdot r \, \frac{d\varphi}{d\xi} = G \cdot r \left\lfloor \frac{dN_1}{d\xi}, \frac{dN_2}{d\xi} \right\rfloor \cdot \begin{Bmatrix} q_2 \\ q_4 \end{Bmatrix}_T =$$

$$= G \cdot \frac{q_4 - q_2}{l_e} \cdot r \quad , \quad \tau_{max} = \frac{Gd}{2} \cdot \frac{q_4 - q_2}{l_e}$$

$$\sigma_{eqv}^{max} = \sqrt{\sigma_x^2 + 3\tau_{max}^2}$$